Journal of Algebra 330 (2011) 177–187

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the derived category of the Cayley plane

L. Manivel

Institut Fourier, Laboratoire de Mathématiques, UMR 5582 UJF-CNRS, BP 74, 38402 St Martin d'Hères Cedex, France

ARTICLE INFO

Article history:
Received 23 July 2009
Available online 21 January 2011
Communicated by Peter Littelmann

Keywords:
Homogeneous space
Octonion
Cayley plane
Derived category
Exceptional bundle

ABSTRACT

We describe a maximal exceptional collection on the Cayley plane, the minimal homogeneous projective variety of E_6. This collection consists in a sequence of 27 irreducible homogeneous bundles.

1. Introduction

The derived categories of coherent sheaves on rational complex homogeneous spaces have been investigated by lots of people since the seminal works of Beilinson [Be] and Kapranov [Ka]. In the cases which are now understood (see [Bo,Sa] for an overview), the derived category of a rational homogeneous space X is generated by an exceptional collection of homogeneous vector bundles. Since the classes in K-theory, of bundles from an exceptional collection, are independent, and since the K-theory of X is known to be a free-module over the classes of the structure sheaves of the Schubert varieties, the length of such a collection has to be equal to the topological Euler characteristic of X.

In this paper we consider the case where $X = \mathbb{OP}^2$ is the Cayley plane, the closed orbit in the projectivization of the minimal representation of the simply connected exceptional group of type E_6. This is a very remarkable variety of dimension 16, which has attracted great interest in complex projective geometry as the last *Severi variety* [LV,Za]. The name and notation we use originate from the fact that it can be considered as a projective plane over the Cayley algebra $\mathbb{O}$ of octonions.

The topological Euler characteristic of $\mathbb{OP}^2$ is 27. We exhibit in Theorem 2 a strongly exceptional collection consisting of 27 homogeneous vector bundles. Apart from line bundles, these bundles are constructed from the minimal homogeneous bundle on $\mathbb{OP}^2$, whose rank is ten, and its symmetric

E-mail address: Laurent.Manivel@ujf-grenoble.fr.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.12.011

powers of degree two and three. This exceptional collection does certainly generate the derived category of the Cayley plane, but we have not been able to prove this property, which remains as a conjecture.

2. The Cayley plane

Let **O** denote the normed algebra of (real) octonions (see e.g. [Ba]), and let $\mathbb{O}$ be its complexification. The space

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix},\ c_i \in \mathbf{C},\ x_i \in \mathbb{O} \right\} \cong \mathbf{C}^{27}$$

of $\mathbb{O}$-Hermitian matrices of order 3, is the exceptional simple complex Jordan algebra (with respect to the product $X.Y = \frac{1}{2}(XY + YX)$).

Although the algebra of octonions is neither commutative nor associative, there is a notion of determinant for matrices in $\mathcal{J}_3(\mathbb{O})$ [Ba, 3.4]. Moreover, the group $SL_3(\mathbb{O})$ consisting of invertible transformations of $\mathcal{J}_3(\mathbb{O})$ preserving this cubic form is the adjoint group of type E_6. The action of E_6 on the projectivization $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6-orbit. These three orbits can be viewed as the (projectivized) sets of matrices of rank three, two, and one respectively.

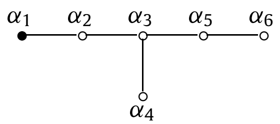
The closed orbit, i.e. the (projectivization of) the set of rank one matrices, is called the *Cayley plane* and denoted $\mathbb{OP}^2$. It can be defined by the quadratic equation

$$X^2 = \mathrm{trace}(X)X, \quad X \in \mathcal{J}_3(\mathbb{O}),$$

or as the closure of the affine cell

$$\mathbb{OP}^2_0 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{pmatrix},\ x, y \in \mathbb{O} \right\} \cong \mathbf{C}^{16}.$$

Since the Cayley plane is a closed orbit of E_6, it can also be identified with the quotient of E_6 by a parabolic subgroup, namely the maximal parabolic subgroup P_1 defined by the simple root α_1 in the notation below. The semi-simple part of this maximal parabolic is isomorphic to Spin_{10}.



If we denote by V_ω the irreducible E_6-module with highest weight ω, we have $\mathcal{J}_3(\mathbb{O}) \simeq V_{\omega_1}$. This is a *minuscule* module, meaning that its weights with respect to any maximal torus of E_6, are all conjugate under the action of the Weyl group $W(E_6)$. For more details, see [LM,IM].

Note that the Dynkin diagram of type E_6 has an obvious symmetry of order two, which accounts for the duality between irreducible modules. For example, the dual module of V_{ω_1} is V_{ω_6}.

3. Homogeneous bundles on the Cayley plane

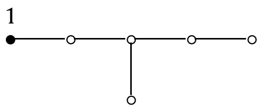
3.1. Irreducible homogeneous bundles

The category of homogeneous bundles on a rational homogeneous variety G/P is equivalent to the category Mod_P of P-modules. Recall that P has a non-trivial decomposition $P = LP^u$, where

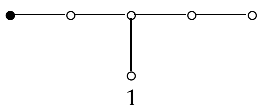
P^u denotes the unipotent radical and L a Levi factor. Since P^u is non-trivial, P is not reductive, and P-modules are not completely reducible in general. Irreducible P-modules have a trivial action of P^u, so that they are completely determined by their L-module structure. Since L is reductive, its irreducible modules are well understood: they are uniquely determined by their highest weight ω, which can be any L-dominant weight of G (recall that the weight lattices of L and G coincide). We denote by $\mathcal{E}_\omega$ the corresponding irreducible homogeneous vector bundles on G/P. By the Borel–Weil theorem, $H^0(G/P, \mathcal{E}_\omega) = V_\omega^\vee$ if ω is dominant, and otherwise $H^0(G/P, \mathcal{E}_\omega) = 0$.

For the Cayley plane $\mathbb{OP}^2 = E_6/P_1$, a Levi factor L of P_1, modded out by its one-dimensional center, is a copy of Spin_{10}. An L-dominant weight ω is a linear combination $\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + a_5\omega_5 + a_6\omega_6$ of the fundamental weights of $\mathfrak{e}_6$, with $a_2, \dots, a_6 \geqslant 0$. We can encode ω by the Dynkin diagram of E_6, where the node corresponding to the fundamental weight ω_i is labeled a_i.

Example 1. The weight $\omega = -\omega_1$ defines a character of L. So $\mathcal{E}_{-\omega_1}$ is just a line bundle, the negative generator of the Picard group. The dual bundle $\mathcal{E}_{\omega_1}$ defines the embedding of $\mathbb{OP}^2$ in $\mathbb{P}V_{\omega_1} = \mathbb{P}\mathcal{J}_3(\mathbb{O})$ and will be denoted $\mathcal{O}_{\mathbb{OP}^2}(1)$.

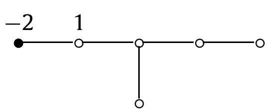
$\mathcal{O}_{\mathbb{OP}^2}(1) \simeq$ 

Example 2. The weight that defines the tangent bundle of $\mathbb{OP}^2$ is the highest root of $\mathfrak{e}_6$, which is also the dominant weight defining the adjoint representation. Note that the corresponding representation of Spin_{10} is one of the half-spin representations, which has dimension sixteen, as the Cayley plane.

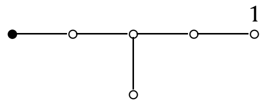
$T_{\mathbb{OP}^2} \simeq$ 

The Borel–Weil theorem yields that $H^0(\mathbb{OP}^2, T_{\mathbb{OP}^2}) = \mathfrak{e}_6$, as expected.

Since the two half-spin representations of Spin_{10} are dual one of the other, one could expect that the weight defining the cotangent bundle of $\mathbb{OP}^2$ be ω_2. This is not exactly true: the defining weight is $\omega_2 - 2\omega_1$, where substracting ω_1 amounts, at the level of bundles, to twisting by $\mathcal{O}_{\mathbb{OP}^2}(-1)$. To check this, one needs to remember that if an irreducible L-module has highest weight ω, then its lowest weight is $w_0^L(\omega)$, where w_0^L denotes the longest element of the Weyl group $W(L)$ of $L \simeq \mathrm{Spin}_{10} \times \mathbb{C}^*$, and then the highest weight of the dual module is $-w_0^L(\omega)$. But this weight must be computed inside the weight lattice of $\mathfrak{e}_6$, on which $W(L)$ acts naturally since it is embedded in $W(E_6)$. And the result of this computation will be what it would be in the weight lattice of Spin_{10}, only up to extra multiples of ω_1. More explicitly, let σ be the transposition of 2 and 4. Then $w_0^L(\omega_i) = -\omega_{\sigma(i)} + a_i\omega_1$ for $2 \leqslant i \leqslant 6$, and a computation shows that $a_6 = 1$, $a_5 = a_4 = a_2 = 2$ and $a_3 = 3$.

$\Omega^1_{\mathbb{OP}^2} \simeq$ 

Example 3. The minimal non-trivial representation of Spin_{10} is the vector representation. This implies that up to line bundles, the irreducible homogeneous bundle defined by ω_6 has minimal rank, equal to ten. We denote it by $\mathcal{S}$.

$\mathcal{S} \simeq$ 

The vector representation of Spin_{10} is self-dual. For the reasons explained above, this does not quite imply that $\mathcal{S}$ be self-dual, but this must be the case up to a twist by a line bundle. Since $a_6 = 1$ we have $\mathcal{S}^\vee = \mathcal{S}(-1)$.

The geometric interpretation of $\mathcal{S}$ is the following. By the Borel–Weil theorem, we have $H^0(\mathbb{OP}^2, \mathcal{S}) = V_{\omega_6}^\vee = V_{\omega_1} = \mathcal{J}_3(\mathbb{O})$. An irreducible homogeneous bundle with non-trivial sections is generated by global sections, so dualizing the evaluation map we get an injection

$$\mathcal{S}^\vee \hookrightarrow \mathcal{J}_3(\mathbb{O})^\vee \otimes \mathcal{O}_{\mathbb{OP}^2}.$$

This map identifies each fiber of $\mathcal{S}^\vee$ with the linear span of an $\mathbb{O}$-line, a maximal quadric in the dual Cayley plane $\mathbb{OP}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})^\vee$, see [LM]. (Note that the Cayley plane and its dual are isomorphic, but only non-canonically: this reflects the fact that the order two symmetry of the Dynkin diagram can only be realized as an outer automorphism of E_6.) In particular the presence of this maximal quadric reflects the fact that the isomorphism $\mathcal{S}^\vee = \mathcal{S}(-1)$ is symmetric, hence it gives a map

$$Sym^2\mathcal{S} \to \mathcal{O}_{\mathbb{OP}^2}(1).$$

Definition. Let $\mathcal{S}_2$ be the kernel of the map $Sym^2\mathcal{S} \to \mathcal{O}_{\mathbb{OP}^2}(1)$. Since the symmetric square of the vector representation of Spin_{10} is, up to the trivial factor defined by the quadratic form, irreducible, $\mathcal{S}_2$ is an irreducible vector bundle, with highest weight $2\omega_6$.

The quadratic map $Sym^2\mathcal{S} \to \mathcal{O}_{\mathbb{OP}^2}(1)$ induces a cubic map $Sym^3\mathcal{S} \to \mathcal{S}(1)$. Let $\mathcal{S}_3$ be the kernel of this cubic map. This is the irreducible bundle with highest weight $3\omega_6$.

3.2. Bott's theorem

The fundamental tool for computing the cohomology of vector bundles on homogeneous spaces is Bott's theorem, which extends the Borel–Weil theorem for global sections to higher cohomology groups.

Consider on G/P an irreducible vector bundle $\mathcal{E}_\omega$. We have seen that it has non-trivial global sections exactly when ω is dominant. In general, let ρ denote the sum of the fundamental weights, and consider the weight $\omega + \rho$. This weight is *singular* if there exists a root α such that $\langle \omega + \rho, \alpha^\vee \rangle = 0$ (equivalently, $\omega + \rho$ is fixed by the simple reflection s_α). Otherwise, there exists a unique w in the Weyl group such that $w(\omega + \rho)$ is strictly dominant, and then $w(\omega + \rho) - \rho$ is dominant. Let $\ell(w)$ denote the length of w.

Theorem 1 *(Bott's theorem). If $\omega + \rho$ is singular, then $\mathcal{E}_\omega$ is acyclic. Otherwise, there is a unique $w \in W(E_6)$ such that $w(\omega + \rho)$ is strictly dominant. Then*

$$H^{\ell(w)}(G/B, \mathcal{E}_\omega) = V^\vee_{w(\omega+\rho)-\rho},$$

and the other cohomology groups of $\mathcal{E}_\omega$ vanish.

Remark. To check whether the weight $\omega + \rho$ is singular or not, we can proceed as follows. If $\omega + \rho$ is not dominant, one of its coefficients on the basis of fundamental weights, say on ω_i, must be negative. Then we apply the simple reflection s_{α_i}, in order to cross the hyperplane orthogonal to $\alpha_i^\vee$. Not that since E_6 is simply laced, this simply amounts to changing the (negative) coefficient of ω_i into its opposite, and adding it to the coefficients of the fundamental weights connected to ω_i in the Dynkin diagram. Iterating this procedure, we will eventually get a weight with some zero coefficient, which will imply that $\omega+\rho$ is singular, or get a strictly dominant weight which will be the representative $w(\omega+\rho)$ of the $W(E_6)$-orbit of $\omega+\rho$ in the interior of the dominant chamber. In the latter case, the number of applications of these simple reflections is nothing but the length $\ell(w)$ of w, which is the degree of the only non-zero cohomology group of $\mathcal{E}_\omega$.

4. Exceptional sequences

4.1. Exceptional bundles

Recall that an object $\mathcal{F}$ of the derived category of coherent sheaves on a variety X is *exceptional* if $RHom(\mathcal{F},\mathcal{F})=\mathbb{C}$. If $\mathcal{F}$ is represented by a vector bundle F on X, this means that

$$H^i\big(X, End(F)\big)=\delta_{i,0}\mathbb{C}.$$

Proposition 1. *The homogeneous bundles $\mathcal{S},\mathcal{S}_2,\mathcal{S}_3$ on $\mathbb{OP}^2$ are exceptional.*

Proof. If U denotes the vector representation of Spin_{10}, we know that $\bigwedge^2 U$ is an irreducible (and even fundamental) representation, and that Sym^2U splits into a trivial factor generated by the invariant quadratic form, and an irreducible summand. At the level of bundles, since $\mathcal{S}^\vee=\mathcal{S}(-1)$, this implies that

$$End(\mathcal{S})=\mathcal{E}_{\omega_5}(-1)\oplus\mathcal{O}_{\mathbb{OP}^2}\oplus\mathcal{S}_2(-1).$$

The bundle $\mathcal{E}_{\omega_5}(-1)$ has highest weight $\omega=\omega_5-\omega_1$. Since $\omega+\rho=\omega_2+\omega_3+\omega_4+2\omega_5+\omega_6$ is orthogonal to $\alpha_1^\vee$, $\omega+\rho$ is singular. By Bott's theorem we conclude that $\mathcal{E}_{\omega_5}(-1)$ is acyclic. For exactly the same reason $\mathcal{S}_2(-1)$ is also acyclic. We conclude that

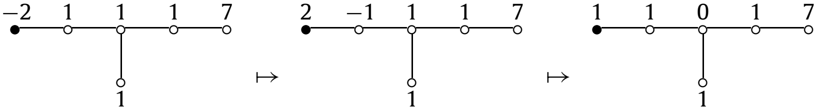
$$H^i\big(\mathbb{OP}^2, End(\mathcal{S})\big)=H^i\big(\mathbb{OP}^2,\mathcal{O}_{\mathbb{OP}^2}\big)=\delta_{i,0}\mathbb{C}.$$

So $\mathcal{S}$ is exceptional.

We proceed similarly with the other two bundles. First observe that $\mathcal{S}_2^\vee=\mathcal{S}_2(-2)$ and $\mathcal{S}_3^\vee=\mathcal{S}_3(-3)$. Using e.g. LiE [LiE] to compute tensor products of representations of Spin_{10}, we get the decompositions:

$$\begin{aligned}
End(\mathcal{S}_2)&=\mathcal{E}_{4\omega_6}(-2)\oplus\mathcal{E}_{\omega_5+2\omega_6}(-2)\oplus\mathcal{E}_{2\omega_5}(-2)\oplus\mathcal{E}_{2\omega_6}(-1)\oplus\mathcal{E}_{\omega_5}(-1)\oplus\mathcal{O}_{\mathbb{OP}^2},\\
End(\mathcal{S}_3)&=\mathcal{E}_{6\omega_6}(-3)\oplus\mathcal{E}_{\omega_5+4\omega_6}(-3)\oplus\mathcal{E}_{2\omega_5+2\omega_6}(-3)\oplus\mathcal{E}_{3\omega_5}(-3)\oplus\mathcal{E}_{4\omega_6}(-2)\\
&\qquad\oplus\mathcal{E}_{\omega_5+2\omega_6}(-2)\oplus\mathcal{E}_{2\omega_5}(-2)\oplus\mathcal{E}_{2\omega_6}(-1)\oplus\mathcal{E}_{\omega_5}(-1)\oplus\mathcal{O}_{\mathbb{OP}^2}.
\end{aligned}$$

Our claim amounts to the acyclicity of all the non-trivial vector bundles in these decompositions, hence, by Bott's theorem, to the singularity of all the corresponding weights, once we have added ρ. We use the remark after Bott's theorem above. Consider for example $\mathcal{E}_{6\omega_6}(-3)$, whose highest weight is $6\omega_6-3\omega_1$. After adding ρ, we get successively, applying s_{α_1} and s_{α_2}:



Since there is a zero label on the rightmost diagram, we conclude that $\mathcal{E}_{6\omega_6}(-3)$ is acyclic. Proceeding in the same way with the other bundles, we conclude the proof. $\square$

Remark. Observe that the irreducible vector bundle $\bigwedge^2 \mathcal{S} = \mathcal{E}_{\omega_5}$ is *not* exceptional. Indeed, if U is again the vector representation of Spin_{10}, $\bigwedge^2 U \otimes \bigwedge^2 U$ contains $\bigwedge^4 U$, which is an irreducible (but not fundamental) representation, contained in the tensor product of the two half-spin representations. This implies that $End(\bigwedge^2 \mathcal{S})$ contains $\mathcal{E}_{\omega_2+\omega_4}(-2)$, which is not acyclic. Indeed, $s_{\alpha_1}(\omega_2 + \omega_4 - 2\omega_1 + \rho) = \omega_4 + \rho$ is strictly dominant, hence

$$H^1\big(\mathbb{OP}^2, \mathcal{E}_{\omega_2+\omega_4}(-2)\big) = V_{\omega_4}^\vee = \mathfrak{e}_6,$$

the latter equality being due to the fact that the highest root of $\mathfrak{e}_6$ is ω_4.

4.2. A maximal exceptional sequence

Recall that an exceptional sequence of sheaves on a projective variety X is a sequence $\mathcal{F}_1, \dots, \mathcal{F}_m$ of exceptional sheaves such that

$$Ext^q(\mathcal{F}_i, \mathcal{F}_j) = 0 \quad \forall q \geqslant 0,\ \forall i > j.$$

It is strongly exceptional if moreover

$$Ext^q(\mathcal{F}_i, \mathcal{F}_j) = 0 \quad \forall q > 0,\ \forall i \leqslant j.$$

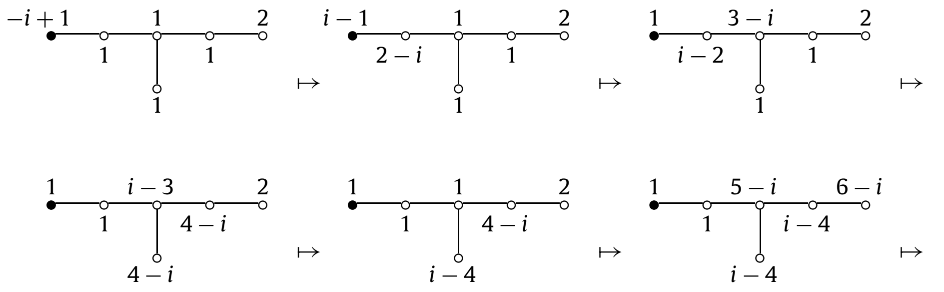
Since $\mathbb{OP}^2$ has index 12, it follows from the Kodaira vanishing theorem that the sequence

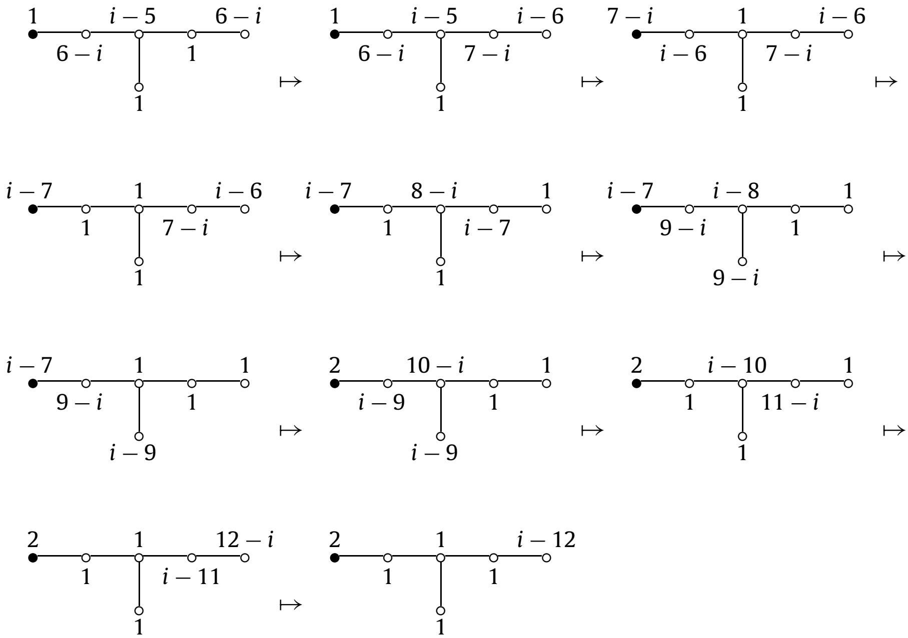
$$\mathcal{O}_{\mathbb{OP}^2}, \mathcal{O}_{\mathbb{OP}^2}(1), \dots, \mathcal{O}_{\mathbb{OP}^2}(10), \mathcal{O}_{\mathbb{OP}^2}(11)$$

is strongly exceptional. On the other hand, it is easy to see that the classes in K-theory of the members of an exceptional sequence are linearly independent (see [Bo]). For rational homogeneous spaces, the K-theory is a free $\mathbb{Z}$-module admitting for basis the classes of the structure sheaves of the Schubert varieties. The length of a maximal exceptional sequence is expected to coincide with the rank of the K-theory, that is, the number of Schubert classes, which is also the topological Euler characteristic of the variety. For the Cayley plane this number is equal to 27, so we expect to be able to enlarge the preceding exceptional sequence of line bundles. For this we will use the exceptional bundles $\mathcal{S}, \mathcal{S}_2$ and $\mathcal{S}_3$, and will apply Bott's theorem again and again.

Lemma 1. *The bundle $\mathcal{S}(-i)$ is acyclic for $1 \leqslant i \leqslant 12$.*

Proof. We play the same game as above, starting with the weight $\omega_6 - i\omega_1 + \rho$. At each step, the weight we get either has a zero coefficient, in which case the game stops and we conclude that we started with a singular weight, or there is a negative coefficient and we apply the corresponding simple reflexion. This goes as follows:





This concludes the proof. □

Note that for $i = 13$ we finally get the strictly dominant weight $\omega_1 + \rho$. Since we needed to apply 16 simple reflexions, we conclude by Bott's theorem that

$$H^{16}\big(\mathbb{OP}^2, \mathcal{S}(-13)\big) = V_{\omega_1}^{\vee}.$$

But by Serre duality, $H^{16}(\mathbb{OP}^2, \mathcal{S}(-13))$ is dual to $H^0(\mathbb{OP}^2 2, \mathcal{S}^{\vee}(1)) = H^0(\mathbb{OP}^2, \mathcal{S})$ which, by Borel–Weil, is $V_{\omega_6}^{\vee} \simeq V_{\omega_1}$. This is a way to check that the computation above, and those of the same type that will follow, are indeed correct.

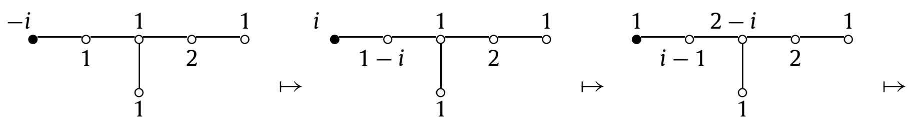
The same statement holds for our two other exceptional bundles:

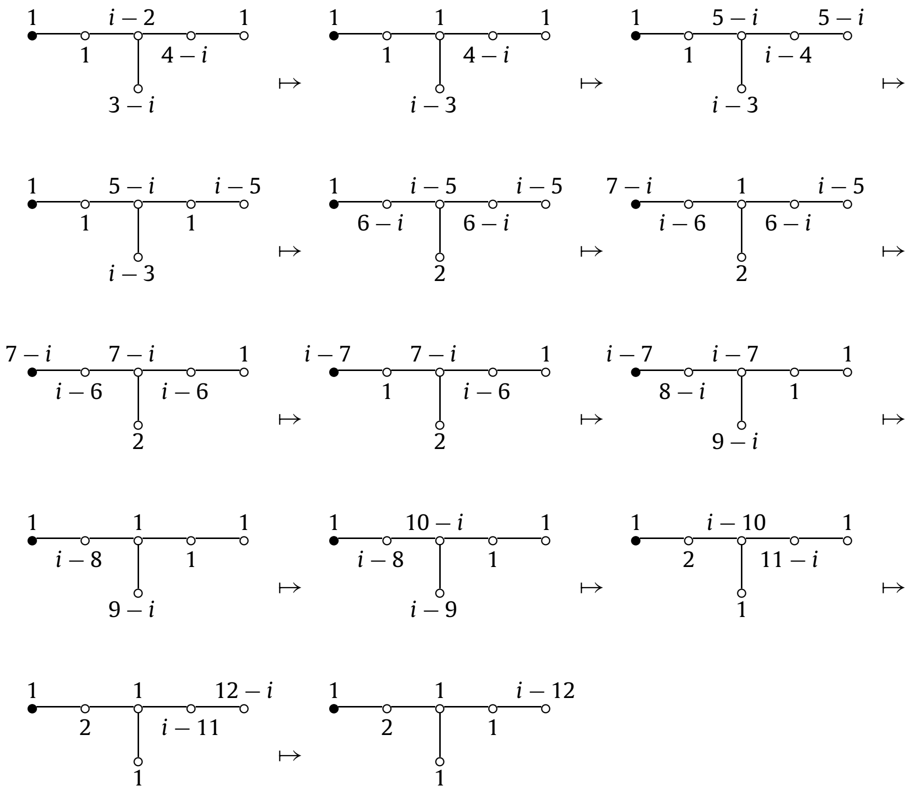
Lemma 2. *$\mathcal{S}_2(-i)$ and $\mathcal{S}_3(-i)$ are acyclic for $1 \leqslant i \leqslant 12$.*

Now consider their endomorphism bundles:

Lemma 3. *$End(\mathcal{S})(-i)$ is acyclic for $1 \leqslant i \leqslant 11$.*

Proof. We have seen that $End(\mathcal{S}) = \mathcal{S}_2(-1) \oplus \mathcal{O}_{\mathbb{OP}^2 2} \oplus \mathcal{E}_{\omega_5}(-1)$. We already know that $\mathcal{S}_2(-i-1)$ and $\mathcal{O}_{\mathbb{OP}^2}(-i)$ are acyclic for $1 \leqslant i \leqslant 11$. There remains to treat the case of $\mathcal{E}_{\omega_5}(-i-1)$, which we do as above. After adding ρ to $\omega_5 - (i+1)\omega_1$, we get successively:

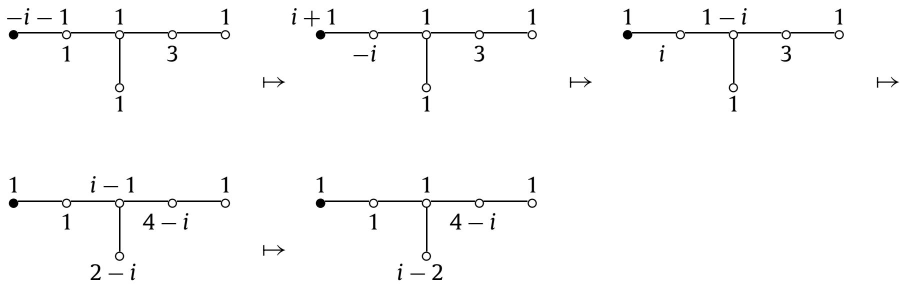




This concludes the proof. □

Lemma 4. *$End(\mathcal{S}_2)(-i)$ is acyclic for $1 \leqslant i \leqslant 2$.*

Proof. We have seen how to decompose $End(\mathcal{S}_2)$ into irreducible bundles. We need to apply Bott's theorem to each component. Consider for example the component $\mathcal{E}_{2\omega_5}(-2)$. After twisting by $\mathcal{O}_{\mathbb{OP}^2}(-i)$ and adding ρ to the corresponding weight, we get:



For $i = 1, 2$ we get singular weights, as claimed. But note that for $i = 3$, the last weight above is ρ, so that $H^3(\mathbb{OP}^2, \mathcal{E}_{2\omega_5}(-5)) = \mathbb{C}$. In particular $End(\mathcal{S}_2)(-3)$ is not acyclic. Examining the other components we can easily complete the proof that $End(\mathcal{S}_2)(-1)$ and $End(\mathcal{S}_2)(-2)$ are both acyclic. □

In a completely similar way, we check that:

Lemma 5. *$End(\mathcal{S}_3)(-1)$ is acyclic.*

Lemma 6. *$\mathcal{S}_2 \otimes \mathcal{S}(-i-1)$ is acyclic for $1 \leqslant i \leqslant 12$.*

Proof. Use the decomposition, that we obtain e.g. using LiE,

$$\mathcal{S}_2 \otimes \mathcal{S} = \mathcal{S}_3 \oplus \mathcal{S}(1) \oplus \mathcal{E}_{\omega_5+\omega_6}.$$

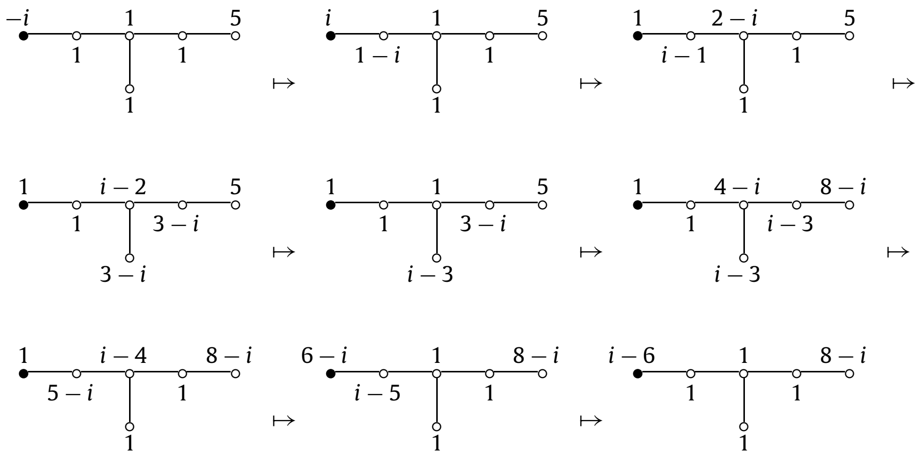
The first two factors have already been considered. The third one is treated in the same way. $\square$

Lemma 7. *$\mathcal{S}_3 \otimes \mathcal{S}(-i-1)$ is acyclic for $1 \leqslant i \leqslant 6$.*

Proof. Here the relevant decomposition is

$$\mathcal{S}_3 \otimes \mathcal{S} = \mathcal{E}_{4\omega_6} \oplus \mathcal{S}_2(1) \oplus \mathcal{E}_{\omega_5+2\omega_6}.$$

The most limiting term is the first one, since it gives rise to the sequence:



For $i = 1, \ldots, 6$ we get singular weights, as claimed, but for $i = 7$ the last weight above is ρ. We can therefore conclude that $H^8(\mathbb{OP}^2, \mathcal{E}_{\omega_5+2\omega_6}(-8)) = \mathbb{C}$. Therefore $\mathcal{S}_3 \otimes \mathcal{S}(-8)$ is not acyclic. We conclude the proof by checking the last component. $\square$

Lemma 8. *$\mathcal{S}_3 \otimes \mathcal{S}_2(-i-2)$ is acyclic for $1 \leqslant i \leqslant 2$.*

Proof. We have the decomposition:

$$\mathcal{S}_3 \otimes \mathcal{S}_2 = \mathcal{E}_{5\omega_6} \oplus \mathcal{E}_{\omega_5+3\omega_6} \oplus \mathcal{E}_{2\omega_5+\omega_6} \oplus \mathcal{E}_{\omega_5+\omega_6}(1) \oplus \mathcal{S}_3(1) \oplus \mathcal{S}(2).$$

The last three terms have already been considered. Among the first three, the most limiting one is the third one, which contributes non-trivially for $i = 3$. But for $i = 1, 2$ all the factors are acyclic. $\square$

We can now prove our main result.

Theorem 2. *The following sequence, of length 27, of vector bundles on the Cayley plane $\mathbb{OP}^2$,*

$$\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \mathcal{S}(1), \mathcal{O}(2), \mathcal{S}(2), \mathcal{O}(3), \mathcal{S}(3), \mathcal{O}(4),$$
$$\mathcal{S}_2(3), \mathcal{S}(4), \mathcal{S}_3(3), \mathcal{O}(5), \mathcal{S}_2(4), \mathcal{S}(5), \mathcal{S}_3(4), \mathcal{O}(6), \mathcal{S}_2(5),$$
$$\mathcal{S}(6), \mathcal{O}(7), \mathcal{S}(7), \mathcal{O}(8), \mathcal{S}(8), \mathcal{O}(9), \mathcal{S}(9), \mathcal{O}(10), \mathcal{O}(11)$$

is a maximal strongly exceptional collection.

Proof. This follows from the previous lemmas. Start with the exceptional collection $\mathcal{O}, \dots, \mathcal{O}(11)$. By Lemma 3, $\mathcal{S}, \mathcal{S}(1), \dots, \mathcal{S}(9)$ is also an exceptional collection. According to Lemma 1, we have $Hom(\mathcal{O}(i), \mathcal{S}(j)) = 0$ for $j < i \leqslant j + 12$. Moreover, since $\mathcal{S}^\vee = \mathcal{S}(-1)$, $Hom(\mathcal{S}(j), \mathcal{O}(i)) = Hom(\mathcal{O}(j+1), \mathcal{S}(i)) = 0$ for $i \leqslant j \leqslant i + 11$. This implies that the sequence

$$\mathcal{O}, \mathcal{S}, \mathcal{O}(1), \mathcal{S}(1), \mathcal{O}(2), \mathcal{S}(2), \mathcal{O}(3), \mathcal{S}(3), \mathcal{O}(4), \mathcal{S}(4), \mathcal{O}(5),$$
$$\mathcal{S}(5), \mathcal{O}(6), \mathcal{S}(6), \mathcal{O}(7), \mathcal{S}(7), \mathcal{O}(8), \mathcal{S}(8), \mathcal{O}(9), \mathcal{S}(9), \mathcal{O}(10), \mathcal{O}(11)$$

is an exceptional collection. On the other hand, Lemmas 1, 4, 5 and 8 imply that

$$\mathcal{S}_2(3), \mathcal{S}_3(3), \mathcal{S}_2(4), \mathcal{S}_3(4), \mathcal{S}_2(5)$$

is also an exceptional collection. There remains to “insert” this collection inside the previous one. The compatibility conditions are the following. For $\mathcal{S}_2$, Lemmas 2, 6 and the fact that $\mathcal{S}_2^\vee = \mathcal{S}_2(-2)$ imply that we must respect the orderings $\mathcal{O}(k) \cdots \mathcal{S}_2(j) \cdots \mathcal{O}(i)$ and $\mathcal{S}(k) \cdots \mathcal{S}_2(j) \cdots \mathcal{S}(i)$ with $j - 10 \leqslant k \leqslant j + 1$ and $j + 1 \leqslant i \leqslant j + 12$. Concerning $\mathcal{S}_3$, Lemmas 2, 7 and the fact that $\mathcal{S}_3^\vee = \mathcal{S}_3(-3)$ imply that we must respect the orderings $\mathcal{O}(k) \cdots \mathcal{S}_3(j) \cdots \mathcal{O}(i)$ with $j - 9 \leqslant k \leqslant j + 2$ and $j + 1 \leqslant i \leqslant j + 12$, and $\mathcal{S}(k) \cdots \mathcal{S}_3(j) \cdots \mathcal{S}(i)$ with $j - 4 \leqslant k \leqslant j + 1$ and $j + 1 \leqslant i \leqslant j + 6$. The collection of the theorem is compatible with these requirements. $\square$

Finally the fact that this collection is strongly exceptional is quite straightforward. Indeed, if $i < j$ and E_i, E_j are the corresponding bundles of the collection, then in most cases $End(E_i, E_j)$ decomposes as a sum of irreducible vector bundles $\mathcal{E}_\omega$ defined by a highest weight ω which is dominant. In this case it is an immediate consequence of Bott's theorem that the higher cohomology groups vanish. Another possibility is that ω has coefficient -1 on ω_1, and then $\mathcal{E}_\omega$ is acyclic. The remaining cases are only of three types, $E_i = \mathcal{S}(k)$ and $E_j = \mathcal{S}_3(k-1)$, or $E_i = \mathcal{S}_2(k)$ and $E_j = \mathcal{S}_3(k)$, or $E_i = \mathcal{S}_3(k)$ and $E_j = \mathcal{S}_2(k+1)$. In these cases the coefficient of ω on ω_1 can be -2, but then the coefficient on ω_3 is zero, and the acyclicity follows immediately.

Of course we expect this maximal exceptional collection to be full, i.e. to generate the derived category of the Cayley plane. This would follow from Conjecture 9.1 and its Corollary 9.3 in [Ku3], but we have not been able to prove it.

A possible strategy would be to find a covering family of subvarieties, possibly of small codimension, for which we already have a good understanding of the derived category. This was the strategy used in [Ku1] for an inductive treatment of Grassmannians of lines. In the Cayley plane, there are at least two natural candidates. The first one is the family of $\mathbb{O}$-lines, that is, of eight-dimensional quadrics parametrized by the dual Cayley plane. The other one is the family of copies of the spinor variety of Spin_{10} in $\mathbb{OP}^2$. Indeed, the union of lines in the Cayley plane passing through a given point is known to be a cone over this spinor variety [LM], which we can recover by taking hyperplane sections not containing the given point. These spinor varieties have codimension six, and their derived category is described in [Ku2, 6.2]. But in both cases the codimension is already sufficiently big to make this strategy difficult to implement concretely.

References

[Ba] J. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2002) 145–205.

[Be] A.A. Beilinson, Coherent sheaves on P^n and problems of linear algebra, Funktsional. Anal. i Prilozhen. 12 (3) (1978) 68–69.

[Bo] Ch. Böhning, Derived categories of coherent sheaves on rational homogeneous manifolds, Doc. Math. 11 (2006) 261–331.

[IM] A. Iliev, L. Manivel, The Chow ring of the Cayley plane, Compos. Math. 141 (2005) 146–160.

[Ka] M.M. Kapranov, On the derived categories of coherent sheaves on some homogeneous spaces, Invent. Math. 92 (1988) 479–508.

[Ku1] A. Kuznetsov, Homological projective duality, Publ. Math. Inst. Hautes Études Sci. 105 (2007) 157–220.

[Ku2] A. Kuznetsov, Exceptional collections for Grassmannians of isotropic lines, Proc. Lond. Math. Soc. 97 (1) (2008) 155–182.

[Ku3] A. Kuznetsov, Hochschild homology and semiorthogonal decompositions, arXiv:0904.4330.

[LM] J.M. Landsberg, L. Manivel, On the projective geometry of homogeneous spaces, Comment. Math. Helv. 78 (2003) 65–100.

[LV] R. Lazarsfeld, A. Van de Ven, Topics in the Geometry of Projective Space, DMV Seminar, vol. 4, Birkhäuser, 1984.

[LiE] LiE, A computer algebra package for Lie group computations, http://young.sp2mi.univ-poitiers.fr/~marc/LiE/.

[Sa] A.V. Samokhin, Some remarks on the derived categories of coherent sheaves on homogeneous spaces, J. Lond. Math. Soc. 76 (1) (2007) 122–134.

[Za] F.L. Zak, Tangents and Secants of Algebraic Varieties, Transl. Math. Monogr., vol. 127, Amer. Math. Soc., 1993.